ON OPERATOR-VALUED FREE CONVOLUTION POWERS.

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Abstract. We give an explicit realization of the η-convolution power of an A-valued distribution, as defined earlier by Anshelevich, Belinschi, Fevrier and Nica. If $\eta : A \to A$ is completely positive and $\eta \geq \mathrm{id}$, we give a short proof of positivity of the η-convolution power of a positive distribution. Conversely, if $\eta \not\geq \mathrm{id}$, and s is large enough, we construct an s-tuple whose A-valued distribution is positive, but has non-positive η-convolution power.

1. Introduction.

In this note, we investigate the question of positivity of η-free convolution powers of an A-valued distribution. Such η-convolution powers were introduced by Anshelevich, Belinschi, Fevrier and Nica in [1], following a question due to Bercovici. For $A = \mathbb{C}$ these correspond to the free convolution powers considered by Nica and Speicher [2]. The main theorem of [1] is a generalization (with a rather complicated proof) of a result from [2]: if μ is a positive A-valued distribution and $\eta : A \to A$ is a completely positive map so that $\eta - \mathrm{id}$ is completely positive, then the convolution power $\mu^{\boxplus\eta}$ is also positive.

In the case that $A = \mathbb{C}$, a simple proof of this theorem exists: for $t > 1$, the convolution powers $\mu^{\boxplus t}$ are realized (after some rescaling) in an explicit way by starting with some random variable X with distribution μ and compressing X to a suitable projection which is free from X (see the appendix to [2] by Voiculescu).

We construct an explicit realization of the distribution of $\mu^{\boxplus\eta}$ as the distribution of v^*Xv, where X has distribution μ, and v is a certain specially constructed element free from X with amalgamation over A (v is a multiple of isometry if $\eta(1)$ is a multiple of 1). Positivity of the distribution of v^*Xv is then immediate. The condition $\eta \geq \mathrm{id}$ appears naturally in the construction of v. Our construction can be viewed as a version of the proof of an explicit realization of $\mu^{\boxplus t}$ using free compression and the Fock space model given in [6].

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In addition, we prove a converse to the theorem of Anshelevich et al: if $\eta - \mathrm{id}$ is not completely positive, for s large enough, there is an s-tuple which has a positive joint A-valued distribution, but so that the η-convolution power of this distribution is not positive.

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1.1. **A-valued distributions and realizability in a C^*-probability space.** We refer the reader to the book [3] for some background on operator-valued free probability theory. Let A be a unital C^*-algebra. Recall that an A-probability space [8, 7] is a unital $*$-algebra $B \supset A$ together with a conditional expectation (i.e, an A-linear map) $E_A^B : B \to A$. For $X \in B$ and a non-commutative monomial $W = a_0 X a_1 X \cdots X a_m$, the value $E_A^B(W)$ is called the non-commutative (A-valued) moment of X; the map $\mu : W \mapsto E_A^B(W)$ is called the (A-valued) distribution of X. We say that an A-valued distribution μ is *positive* if it is possible to find some C^*-algbera B, a positive A-linear map $E_A^B : B \to A$ and $X \in D$ so that the A-valued distributions of X is exactly μ. We say that such an X *realizes* μ.

Positivity is an important property of an A-valued distribution; for $A = \mathbb{C}$ positivity of a distribution corresponds to positivity of a probability measure.

1.2. **Free cumulants ω_k^X.** Associated to any A-valued distribution μ one has a sequence of $\mathbb{C}$-multilinear maps $\omega_k^\mu : A^{k-1} \to A$ called the free cumulants of X (here ω_1^μ is simply an element of A) [8, 7]. Let Q be the universal algebra generated by elements $L^\dagger$, $L_0, L_1, \ldots$ and A subject to the relations:

$$\begin{aligned} L_0 &= \omega_1^\mu \in A \subset Q \\ L^\dagger a_1 L^\dagger a_2 L^\dagger a_3 \cdots L^\dagger a_k L_k &= \omega_{k+1}^\mu(a_1, \ldots, a_k) \in A \subset Q. \end{aligned} \tag{1.1}$$

Finally, let $E_A^Q : Q \to A$ be determined by requiring that $E_A^Q\big|_A = \mathrm{id}$ and that for any non-commutative monomial W in elements of A, $L^\dagger$, L_1, L_2, etc., $E_A^Q(W) = 0$ unless W can be reduced to an element of A using the relations (1.1). Then the sequence $\{\omega_k^\mu\}_{k\geq 1}$ is uniquely determined by the requirement that if we set $Y = L^\dagger + \sum_{k\geq 0} L_k$, the A-valued distribution of Y is μ.

1.3. **η-free convolution powers.** Let μ be an A-valued distribution, and let $\eta : A \to A$ be a linear map. Define a new distribution $\mu^{\boxplus\eta}$ by requiring that its free cumulants are given by $\omega_k^{\mu^{\boxplus\eta}} = \eta \circ \omega_k^\mu$. This

distribution is called, by definition, the η-convolution power of μ (see equation (1.4) in [1]).

2. An explicit realization of the η-convolution powers.

2.1. **Construction of the operator** $v \in (C, E_A^C)$**.** Let A be a C^*-algebra, let $\psi : A \to A$ be a completely-positive map, and let $\eta = \psi + \mathrm{id}$. Let $\mathscr{H}$ be an A, A Hilbert bimodule and $\xi \in \mathscr{H}$ be such that

$$\langle \xi, a\xi \rangle_{\mathscr{H}} = \psi(a).$$

Let $\mathscr{K} = \mathscr{H} \oplus A$ with the inner product $\langle h \oplus a, h' \oplus a' \rangle_{\mathscr{K}} = \langle h, h' \rangle_{\mathscr{H}} + a^* a'$. Then $\mathscr{K}$ is an A, A Hilbert bimodule with the diagonal left and right actions of A. Finally let

$$\mathscr{F} = A \oplus \mathscr{K} \oplus \mathscr{K} \otimes_A \mathscr{K} \oplus \cdots \oplus \mathscr{K}^{\otimes_A n} \oplus \cdots$$

be the full Fock space associated to $\mathscr{K}$ (see [5, 7]). We view $\mathscr{F}$ as an A, A-bimodule using the diagonal left and right actions of A. We'll denote the left action of A on $\mathscr{F}$ by λ.

Let us denote by v the operator

$$v : \mathscr{F} \to \mathscr{F}, \quad \zeta_1 \otimes \cdots \otimes \zeta_n \mapsto (\xi \oplus 1) \otimes \zeta_1 \otimes \cdots \otimes \zeta_n.$$

Then an easy computation shows that

$$v^* \lambda(a) v = \lambda(\eta(a)).$$

Finally, for a bounded adjointable right A-linear operator $T : \mathscr{F} \to \mathscr{F}$, set

$$E_A^C(T) = \langle 0 \oplus 1, T(0 \oplus 1) \rangle_{\mathscr{F}},$$

where we regard $0 \oplus 1 \in \mathscr{K} \subset \mathscr{F}$. Then

$$\begin{aligned} E_A^C(v\lambda(a)v^*) &= \langle 0 \oplus 1, v\lambda(a)v^*(0 \oplus 1) \rangle_{\mathscr{F}} \\ &= \langle 0 \oplus 1, v\ a1 \rangle_{\mathscr{F}} \\ &= \langle 0 \oplus 1, (\xi a \oplus a) \rangle_{\mathscr{F}} = a. \end{aligned}$$

Letting $C = C^*(\lambda(A), v)$, we note that (C, E_A^C) is an A-probability space. We'll also identify A with $\lambda(A)$.

Remark 2.1. (i) Note that $v^* v = v^* \lambda(1) v = \eta(1)$. Thus if $\eta(1) = \alpha 1$ with $\alpha \in \mathbb{R}$, then $\alpha^{-1/2} v$ is an isometry. For general η, v is not an isometry. (ii) In the case that $A = \mathbb{C}$ and $\eta(a) = \lambda a$, $\lambda \in [1, +\infty)$, the conditional expectation E_A^C is non-tracial. Indeed, we have that $E_A^C(vv^*) = E_A^C(v\lambda(1)v^*) = 1$ but $E_A^C(v^*v) = E_A^C(\alpha) = \lambda$.

2.2. **The main result.** Let $X \in (B, E_A^B)$ and assume that X has A-valued distribution μ. We will now compute the A-valued distribution of $\hat{X} = vXv^*$.

Proposition 2.2. *Assume that $\psi : A \to A$ is a completely-positive map, and let $\eta = \psi + \mathrm{id}$ and let $v \in (C, E_A^C)$ be as in §2.1. Let B be a C^*-algebra and $E_A^B : B \to A$ be a positive A-linear map. Let $X = X^* \in B$ having distribution μ. Consider $(M, E_A^M) = (B, E_A^B) *_A (C, E_A^C)$, let $\hat{X} = v^*Xv$, and let $\hat{\mu}$ be the distribution of $\hat{X}$. Then the free cumulants $\omega_k^{\hat{\mu}}$ satisfy:*

$$\omega_k^{\hat{\mu}} = \eta \circ \omega_k^{\mu}. \tag{2.1}$$

In particular, the A-valued distribution of $\hat{X}$ is positive.

Proof. Consider $(N, E_A^N) = (B, E_A^B) *_A (Q, E_A^Q)$, and let $Y = L^\dagger + \sum_{k\geq 0} L_k \in Q$ be as in §1.2. Let $\hat{Y} = v^*Y'v$.

The A-valued distribution of X' is the same as the A-valued distribution of Y'.

Since Q is free from B with amalgamation over A, we may thus assume [4] that $L^\dagger$ and L_k satisfy the relations

$$L^\dagger b_1 L^\dagger b_2 L^\dagger b_3 \cdots L^\dagger b_k L_k = \omega_{k+1}^{\mu}(E_A^C(b_1), \ldots, E_A^C(b_k)), \qquad b_j \in C$$

and moreover for any monomial W in elements of C and $L^\dagger, L_1, L_2, \ldots,$ $E_A^N(W) = 0$

Let $\hat{L}^\dagger = v^*L^\dagger v$ and $\hat{L}_k = v^*L_k v$. Then we have:

$$\begin{aligned}
\hat{L}^\dagger a_1 \hat{L}^\dagger a_2 \cdots \hat{L}^\dagger a_k \hat{L}_k &= v^*L^\dagger v a_1 v^* L^\dagger v a_2 \cdots v^* L^\dagger v a_k v^* L_k v \\
&= v^* \omega_{k+1}^{\mu}(E_A^C(v a_1 v^*), \ldots, E_A^C(v a_k v^*)) v \\
&= v^* \omega_{k+1}^{\mu}(a_1, \ldots, a_k) v \\
&= \eta(\omega_{k+1}^{\mu}(a_1, \ldots, a_k)).
\end{aligned}$$

Moreover, if W is a non-commutative monomial in elements of A and $\hat{L}^\dagger, \hat{L}_1, \hat{L}_2, \ldots,$ then $E_A^N(W) = 0$ unless W can be reduced to an element of A using this relation. It then follows that if $\hat{\mu}$ is the distribution of $\hat{Y}$ (and is the same as the distribution of $\hat{X}$), then its free cumulants are given by

$$\omega_k^{\hat{\mu}} = \eta \circ \omega_k^{\mu}.$$

This completes the proof. □

Theorem 2.3. *Let (B, E_A^B) be an A-probability space and let $X \in B$ be a random variable whose A-valued distribution μ is positive. Let $\eta : A \to A$ be completely-positive map so that $\eta - \mathrm{id}$ is completely positive. Let $\hat{X} = vXv^*$ be as in Proposition 2.2.*

Then the distribution of $\hat{X}$ is the same as that of the η-convolution power [1] $X^{\boxplus\eta}$*; in other words, vXv^* is an explicit realization of $\mu^{\boxplus\eta}$.*

In particular, the A-valued distribution of $\mu^{\boxplus\eta}$ is also positive.

Proof. Let $\psi = \eta - \mathrm{id}$, so that $\eta = \psi + \mathrm{id}$. Let $\hat{X}$ be as in Proposition 2.2. By (2.1) and [1] equation (1.4), the free cumulants of the distribution of $\hat{X}$ and of $\mu^{\boxplus\eta}$ are equal. Thus these A-valued distributions are also equal. But $\hat{X}$ is explicitly realized in a C^*-probabilty space and so its distribution is positive. □

3. A converse.

It is natural to ask whether the condition that $\eta - \mathrm{id}$ be completely-positive is necessary for η-convolution powers to always remain positive (no matter what the initial distribution is). We show that this is indeed the case if one considers joint distributions of all s-tuples.

Theorem 3.1. *Assume that $\eta : A \to A$ is a completely positive map. Then $\eta - \mathrm{id}$ is completely-positive iff for every $s \geq 1$ and every positive A-valued distribution μ of an s-tuple, $\mu^{\boxplus\eta}$ is also positive.*

Proof. There is a natural equivalence between A-valued distributions $\mu^{(X_{ij})}$ of m^2-tuples of variables $(X_{ij})_{i,j=1}^m$ and of the $M_{m\times m}(A)$-valued distribution μ^X of the matrix $X = (X_{ij})$. In fact, one easily obtains that the η-convolution power of $\mu^{(X_{ij})}$ (defined by the requirement that the joint cumulants are composed with η) correspond exactly to the $(\mathrm{id}_m \otimes \eta)$-convolution powers of μ^X. Thus positivity of $\mu^{\boxplus\eta}$ for every A-valued distribution of an s-tuple is equivalent to positivity of $\nu^{\boxplus(\mathrm{id}_m \otimes\eta)}$ for every $M_{m\times m}(A)$-valued distribution of a single variable ν. This completes the proof of one direction of the theorem.

Assume now that there exists integer m and $a \in M_{m\times m}(A)$, $a > 0$ so that $\eta_m(a) - a$ is not positive (here $\eta_m = \mathrm{id} \otimes \eta$). Let ϕ be a state on $M_{m\times m}(A)$ so that $\phi(\eta_m(a) - a) < 0$. Passing from A to the enveloping von Neumann algebra A^{**}, and from a to a spectral projection of a, we may assume that $a \in M_{m\times m}(A^{**})$ is projection and ϕ still satisfies $\phi(\eta_m(a) - a) < -2\kappa < 0$ for some fixed $\kappa > 0$. By replacing ϕ with a convex linear combination with a state that is strictly positive on a we may assume that $\phi(a) > 0$ and sill $\phi(\eta_m(a)) < \phi(a) - \kappa$.

Let $\pi : M_{m\times m}(A^{**}) \to B(H)$ be the GNS construction for ϕ and denote by ξ the associated cyclic vector in H. Let $P \in B(H)$ be the rank one projection onto ξ. Denote by $\hat{A}$ the C^*-algebra generated by $M_{m\times m}(A^{**})$ and P inside $B(H)$.

Choose $\delta > 0$ so that $\delta < \kappa$.

Note that $Tr(aPa) = \phi(a)$. Since aPa is finite-rank, we can find N orthonormal vectors $\xi_j \in H$, $j = 1, \ldots, N$ so that $\xi_1 = \xi$ and $|Tr(aPa) - \sum \langle aPa\xi_j, \xi_j\rangle| < \delta$. Thus

$$\left|\sum \langle aPa\xi_j, \xi_j\rangle - \phi(a)\right| < \delta.$$

Let $\vartheta(x) = \frac{1}{N}\sum\langle x\xi_j, \xi_j\rangle$ be a state on $\hat{A}$. Then $\vartheta(P) = \frac{1}{N}$ and so

$$\left|\frac{\vartheta(aPa)}{\vartheta(P)} - \phi(a)\right| < \delta. \tag{3.1}$$

Let $X \in (B, \psi)$ be a self-adjoint random variable in a $\mathbb{C}$-valued C^*-probability space B, and consider

$$(C, \theta) = (\hat{A}, \vartheta) * (B, \psi).$$

Denote by $E = E^C_{\hat{A}}$ the conditional expectation from C onto $\hat{A}$. If ω_n denotes the n-th scalar-valued cumulant of X, then the $\hat{A}$-valued cumulants of $a^{1/2}Xa^{1/2}$ are given by

$$\omega'_{n+1}(h_1, \ldots, h_n) = \omega_{n+1}\ a\vartheta(ah_1a)\cdots\vartheta(ah_na)a$$

(see [4]) and thus (recalling that $a^2 = a$) the $\hat{A}$-valued cumulants of the η-amplification Y of the distribution of aXa are given by

$$w''_{n+1}(h_1, \ldots, h_n) = \eta_m(a)\ \omega_{n+1}\ \prod\vartheta(ah_ja).$$

This means that the $\hat{A}$-valued cumulants of PYP are given by

$$\begin{aligned}\omega'''_{n+1}(h_1, \ldots, h_n) &= P\eta_m(a)P\cdot\omega_{n+1}\cdot\prod\vartheta(aPh_jPa)\\ &= \phi(\eta_m(a))\omega_{n+1}\cdot P\prod\vartheta(aPh_jPa)\end{aligned}$$

(since $P\eta(a)P = \phi(\eta(a))P$). From this we see that the scalar-valued cumulants of PYP with respect to θ are given by

$$\hat{\omega}_{n+1} = \vartheta(P)\phi(\eta_m(a))\vartheta(aPa)^n\cdot\omega_{n+1}.$$

Let us finally set $Z = \vartheta(aPa)^{-1}PYP$. Then its scalar-valued cumulants are given by

$$\tilde{\omega}_{n+1} = \vartheta(P)\phi(\eta_m(a))\frac{\vartheta(aPa)^n}{\vartheta(aPa)^{n+1}}\omega_n = \frac{\vartheta(P)}{\vartheta(aPa)}\phi(\eta_m(a))\omega_{n+1}.$$

Thus $\tilde{\omega}_n = \lambda\omega_n$ with

$$\lambda = \frac{\phi(\eta_m(a))}{\vartheta(aPa)/\vartheta(P)} < \frac{\phi(a) - \kappa}{\vartheta(aPa)/\vartheta(P)}.$$

By (3.1) and our choice of $\delta < \kappa$, we conclude that

$$\lambda < \frac{\phi(a) - \kappa}{\phi(a) - \delta} < 1.$$

In other words, the $\mathbb{C}$-valued distribution of Z is the same as that of the λ-convolution power of the $\mathbb{C}$-valued distribution of X for some $\lambda < 1$.

Assume now for contradiction that the laws of all $\mathrm{id} \otimes \eta$-convolution powers of $M_{m\times m}(A)$ are positive. Choose $a_k \in M_{m\times m}(A)$ so that $a_k \to a$ weakly and $\sup \|a_k\| < \infty$. Then if we set Y_k to be the η-convlution power of the distribution of $a_k X a_k$ (which is positive by our assumption), and $Z_k = \vartheta(aPa)^{-1} P Y_k P$, then we see that $Z_k \to Z$ in moments. But then positivity of distributions of Z_k implies positivity of the distribution of Z.

To summarize, assuming that the η-convolution power of distribution of every $M_{m\times m}(A)$-valued distribution is positive, we concluded that the law of the scalar-valued distribution of X admits a positive λ-convolution power for some $\lambda < 1$. But this is not always possible: for example, we could start with X having as distribution the sum of two equal point masses; it is known that this distribution admits no λ-convolution power if $\lambda < 1$. $\square$

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